* Agradecimientos: PIA Ciencias Cognitivas, Centro de Investigación en Ciencias Cognitivas, Instituto de Estudios Humanísticos, Universidad de Talca. Fondequip (Programa de Equipamiento Científico y Tecnológico), 2019, código EQM190153.

Exclusive Disjunctions With Three Disjuncts from First-Order Predicate Calculus*

Disyunciones exclusivas con tres términos desde el cálculo de predicados de primer orden

DOI:
doi.org/10.23924/oi.v15i34.629

Miguel López-Astorga
Universidad de Talca, Chile
milopez@utalca.cl
orcid.org/0000-0002-6004-0587

Fecha de recepción: 05/05/2023 • Fecha de aceptación: 29/05/2023

Abstract

First-order predicate logic seems to be incompatible with the way people understand embedded exclusive disjunctions with three disjuncts. In classical logic, an exclusive disjunction with three disjuncts holds when the three disjuncts hold. However, it is hard to note that for people. The theory of mental models can explain this fact. According to that theory, individuals tend to process embedded exclusive disjunctions with three disjuncts intuitively. Thus, they only consider possible situations in which just one of the disjuncts is the case. The present paper tries to explain this problem within first-order predicate logic. The main point is that, in the latter logic, in inferences having, as its first premise, an

Resumen

La lógica de predicados de primer orden parece incompatible con el modo en que las personas comprenden las disyunciones exclusivas embebidas con tres términos. En la lógica clásica, una disyunción exclusiva con tres términos sostiene cuando sus tres términos sostienen. Sin embargo, es difícil para los seres humanos notar eso. La teoría de los modelos mentales puede explicar este hecho. Según dicha teoría, los individuos tienden a procesar las disyunciones exclusivas embebidas con tres términos intuitivamente. Así, solo consideran situaciones posibles en las que únicamente uno de los términos es el caso. Este trabajo trata de explicar este problema desde la lógica de predicados de primer orden. Su punto embedded exclusive disjunction with three disjuncts, and, as its second premise, the first disjunct of that very exclusive disjunction, it is possible to infer none of the other two disjuncts.

Keywords

Classical logic; exclusive disjunction; first-order predicate calculus; reason; theory of mental models.

principal es que, en esta última lógica, en inferencias que tienen, como primera premisa, una disyunción exclusiva embebida con tres términos y, como segunda premisa, el primer término de esa misma disyunción exclusiva, no es posible inferir ninguno de los otros dos términos.

Palabras clave

Cálculo de predicados de primer orden; disyunción exclusiva; lógica clásica; razón; teoría de los modelos mentales.

Introduction

It seems that people do not understand embedded exclusive disjunctions with three disjuncts in the way first-order predicate logic requires. That is what experimental results appear to show (e.g., Khemlani & Johnson-Laird, 2009). The literature reveals that, given a task such as (1), individuals do not respond as, in principle, first-order predicate calculus claims.

(1) Sentences A and B can be neither true nor false at once.
Sentence A: 'you read a book'.
Sentence B: 'either you watch a movie or you play a video game, but you do not do both actions'.
Assume that you read a book. Can you also watch a movie? Can you also play a video game? Can you also both watch a movie and play a video game?

Following first-order predicate logic, the answer to the three questions in (1) must be positive. This is because sentence B is false whenever you watch a movie and you play a video game. However, experiments with different thematic content but the same structure as (1) have been carried out. The results of those experiments indicate that the majority trend is to respond to questions in tasks such as (1) in a negative way (e.g., Khemlani & Johnson-Laird, 2009). It appears that people are not always aware that exclusive disjunctions with two disjuncts require one of the disjuncts to be false. It is possible to give an account of the results from different frameworks. If it is assumed that people do not make inferences in the manner first-order predicate calculus claims, but, for example, in the way the theory of mental models (e.g., Johnson-Laird, 2023) proposes, it is easy to explain participants' answers. It is not difficult to account for the results in tasks such as (1) from logic either. For instance, although the proponents of the theory of mental models do not agree with that (e.g., Khemlani, Hinterecker, & Johnson-Laird, 2017), the explanation by the latter theory can be linked to an approach based on modal logic (e.g., López-Astorga, 2021).

The novelty of this paper is that it will try to explain the results in tasks such as (1) resorting only to first-order predicate calculus. The first section will describe why the usual results in those tasks can be a problem for classical logic. Second, as an example, the way the theory of mental models can remove the problem will be presented. Third, the paper will show that those very results can also be understood from first-order predicate logic.

Exclusivity and disjunctions with three disjuncts

Exclusive disjunctions with three disjuncts have been dealt with since ancient times. An important analysis was offered, for instance, as a part of Stoic logic (e.g., O'Toole & Jennings, 2004). Within classical propositional logic, tasks akin to (1) are a problem because individuals often respond that 'you cannot watch a movie', 'you cannot play a video game', and 'you cannot watch a movie and play a video game at once' (for other examples different from that addressed in this section, see, e.g., Khemlani & Johnson-Laird, 2009). The reasons why this causes difficulties in classical logic are easy to see if the following equivalences are assumed.

$r =_{df}$ you read a book

$w =_{df}$ you watch a movie

$p =_{df}$ you play a video game

If, in addition, 'v(x)' is considered to mean 'the truth value of x', '1' indicates truth, '0' represents falsity, and 'EDF(x, y)' denotes that there is an exclusive disjunctive relation between x and y, in classical propositional logic, it can be said that,

(2) v[EDF(w, p)] = 1 if and only if (from now on, IFF) either v(w) = 1 and v(p) = 0, or v(w) = 0 and v(p) = 1.

On the other hand,

(3) v[EDF(w, p)] = 0 IFF either v(w) = 1 and v(p) = 1, or v(w) =0 and v(p) = 0.

'EDF(w,p)' captures what sentence B in (1) expresses: there is an exclusive disjunctive relation between 'w' and 'p'. Given that sentence A and sentence B cannot hold and cannot be false at the same time, 'EDF[r, EDF(w,p)]' should be the case, which implies what (4) provides.

(4) v{EDF[r, EDF(w, p)]} = 1 IFF either v(r) = 1 and v[EDF(w,p)] = 0, or v(r) = 0 and v[EDF(w,p,)] = 1.

If (4) is true, (5) is true too.

(5) v{EDF[r, EDF(w,p)]} = 1 IFF one of these alternatives is the case:
Alternative 5.1: v(r) = 1, v(w) = 1, and v(p) = 1.
Alternative 5.2: v(r) = 1, v(w) = 0, and v(p) = 0.
Alternative 5.3: v(r) = 0, v(w) = 1, and v(p) = 0.
Alternative 5.4: v(r) = 0, v(w) = 0, and v(p) = 1.

Alternative 5.1 points out that it is possible that 'you read a book', 'you watch a movie', and 'you play a video game'. Actually, it points out that the three propositions can be true at the same time. Therefore, here is a problem for classical logic. Perspectives different from logic can solve the problem. One of those perspectives is, for example, that of the theory of mental models (see also, e.g., Khemlani, Byrne, & Johnson-Laird, 2018).

Exclusivity and disjunctions with three disjuncts: the account from the theory of mental models

A theory such as the theory of mental models can explain the difficulties tasks such as (1) cause. The account from the theory of mental models is to be found in the literature (e.g., Khemlani & Johnson-Laird, 2009). That account will be considered in this section (so, the explanation in this section will be based on the explanation in Khemlani & Johnson-Laird, 2009). It will be described by means of the machinery the most updated version of the theory offers (see also, e.g., Johnson-Laird & Ragni, 2019).

According to the theory of mental models, there are 'conjunctions of possibilities' corresponding to each of the traditional connectives (e.g., Khemlani, Byrne, & Johnson-Laird, 2018). In the case of an exclusive disjunction such as that in sentence B in (1), which is the relevant connective for this paper, the conjunction of possibilities is easy to identify (see also, e.g., Johnson-Laird, Quelhas, & Rasga, 2021). Sentence B in (1) provides that 'either you watch a movie or you play a video game, but you do not do both actions'. Hence, the conjunction is that in (6).

(6) Possible (you watch a movie & you do not play a video game) &
Possible (you do not watch a movie & you play a video game)

The important point the theory of mental models makes in this regard is that, to note all of the clauses in (6), it is necessary a detailed analysis. If sentence B in (1) is addressed only in an intuitive way (i.e., in a quick and without reflection way), the clauses denied in (6) may not be identified. This is because the theory of mental models is a dual-process theory (see also, e.g., Quelhas, Rasga, & Johnson-Laird, 2019). Intuitive thought does not lead to (6), but to (7).

(7) Possible (you watch a movie) &
Possible (you play a video game)

One might think that, given the disjunction with three disjuncts underlying (1), its possibilities should include both the cases with sentence A being true and sentence B being false, and the cases with sentence A being false and sentence B being true. This is hard for two reasons. First, if people only use their intuition, they will forget negations and false situations. Thus, just one more possibility will be added to (7): the possibility corresponding to sentence A in (1). Conjunction of possibilities (8) captures this.

(8) Possible (you read a book) &
Possible (you watch a movie) &
Possible (you play a video game)

Second, if all of the real possibilities are thought, that also implies to consider the scenarios related to sentence A and sentence B that are not allowed. The problem is that, following the theory of mental models (e.g., Barres & Johnson-Laird, 2003), to identify the circumstances that are not acceptable, it is first necessary to recover the situations that are possible, which makes the action even more difficult. In the case of sentence B, that means that, to know what the scenarios in which sentence B is not the case are, it is needed to be aware of (6) before. This is because the scenarios forbidden for sentence B are those missing in (6), that is, those in (9).

(9) Possible (you watch a movie & you play a video game) &
Possible (you do not watch a movie & you do not play a video game)

Taking into account that the inadmissible situation if sentence A in (1) is true is that in (10),

(10) Possible (you do not read a book)

The following conjunctions of possibilities can be built:

The circumstances in which sentence A is true and sentence B is false are those in (11).

(11) Possible (you read a book & you watch a movie & you play a video game) &
Possible (you read a book & you do not watch a movie & you do not play a video game)

Conjunction of possibilities (11) is the result of joining the scenario in which sentence A in (1) is true (i.e., the first possibility or conjunct in (8)) and the possibilities making sentence B in (1) false (i.e., those in (9)).

On the other hand, the situations in which sentence A in (1) is false and sentence B in (1) is true are those in (12).

(12) Possible (you do not read a book & you watch a movie & you do not play a video game) &
Possible (you do not read a book & you do not watch a movie & you play a video game)

Conjunction of possibilities (12) is the result of joining the scenario in which sentence A in (1) is not the case (i.e., that in (10)) and the possibilities in which sentence B in (1) holds (i.e., those in (6)).

So, the final conjunction of possibilities corresponding to (1) consists of the conjunction of (11) and (12), that is, (13).

(13) Possible (you read a book & you watch a movie & you play a video game) &
Possible (you read a book & you do not watch a movie & you do not play a video game) &
Possible (you do not read a book & you watch a movie & you do not play a video game) &
Possible (you do not read a book & you do not watch a movie & you play a video game)

If individuals come to (13), they can note that it is possible that 'you read a book', 'you watch a movie', and 'you play a video game'. The first possibility in (13) shows that the three actions can be carried out at the same time. The inconvenience is that, as pointed out, the machinery of the theory of mental models reveals that to build (13) is hard. It requires complex mental processes and deductive rigor. Given a task such as (1), it is much easier to follow intuition. It can be thought that this is what happens in most occasions. Thereby, people tend to consider just possibilities such as those in (8). When (8) is the conjunction of possibilities taken into account, and, in addition, it is assumed that 'you read a book', neither 'you watch a movie' nor 'you play a video game' can be derived. The two latter sentences cannot be inferred at once either.

This is the explanation from the theory of mental models about the usual results in tasks such as (1) (e.g., Khemlani & Johnson-Laird, 2009). According to the theory, intuition is responsible for those results. This is because, given a task of that kind, for the latter theory, intuition works as described. More accounts are possible. However, for the aims of this paper, the example of the theory of mental models suffices.

Exclusivity and disjunctions with three disjuncts: the account from first-order predicate calculus

The explanation the theory of mental models gives is clear. Besides, it has the advantage that the theory of mental models can account for many other cognitive phenomena (e.g., Byrne & Johnson-Laird, 2020; Espino, Byrne, & Johnson-Laird, 2020; Khemlani & Johnson-Laird, 2022). Accordingly, the explanation can be included in a more general framework trying to describe how human intellectual activity works.

Beyond these facts, a question remains: can first-order predicate logic somehow explain the reasons why individuals do not often respond to tasks such as (1) as classical logic claims? This section will try to argue that first-order predicate calculus can offer those reasons. To show that, first, these predicates must be defined:

$R =_{df}$ to read a book

$W =_{df}$ to watch a movie

$P =_{df}$ to play a video game

Constant 'a' has to be defined too:

$a =_{df}$ you

From these definitions, it can be stated that the logical form of sentence A in (1) is (14).

(14) Ra

Likewise, the logical form of sentence B in (1) can be (15).

(15) $Wa \veebar Pa$

Where '$\veebar$' represents exclusive disjunction.

Nevertheless, (15) can be expressed in other way in classical logic. As in other works in the literature dealing with tasks such as (1) (e.g., López-Astorga, 2014), (15) can be transformed into an inclusive disjunction to which, by means of a conjunction, another well-formed formula is added: a formula indicating that the conjunction of its two disjuncts (which are deemed as conjuncts) is false. This can be made by virtue of (16).

(16) $(X \veebar Y) =_{df} [(X \vee Y) \wedge \neg(X \wedge Y)]$

Where '$\vee$' stands for inclusive disjunction, '$\wedge$' denotes conjunction, and '$\neg$' is the symbol for negation.

Thus, (15) can be transformed into (17).

(17) $(Wa \vee Pa) \wedge \neg(Wa \wedge Pa)$

Formulae (14) and (17) enable to come to the logical form of (1), which can be (18).

(18) $Ra \rightarrow \neg[(Wa \vee Pa) \wedge \neg(Wa \wedge Pa)]$

Where '$\rightarrow$' establishes material conditional relation.

Formula (18) indicates that, if sentence A in (1) is true, sentence B in (1) should be false. This can be expressed the other way round. (19) captures the idea that, if sentence B in (1) is true, sentence A in (1) needs to be false.

(19) $[(Wa \vee Pa) \wedge \neg(Wa \wedge Pa)] \rightarrow \neg Ra$

But it is not necessary to make (19) explicit. (19) derives from (18) in first-order predicate calculus.

Task (1) asks for assuming that 'you read a book', that is, Ra. From Ra and (18), via Modus Ponendo Ponens, (20) can be deduced.

(20) $\neg[(Wa \vee Pa) \wedge \neg(Wa \wedge Pa)]$

Nonetheless, neither Wa nor Pa can be inferred from (20). Hence, from first-order predicate logic, it is justified that participants in tasks such as (1) negatively respond to the questions included in it.

In first-order predicate calculus, at most, a formula such as (20) can be transformed, for example, into a formula such as (21).

(21) $(Wa \vee Pa) \rightarrow (Wa \wedge Pa)$

But, again, it keeps being impossible to deduce 'Wa' or 'Pa'. This can explain why, given a task such as (1), individuals tend to respond that it is possible neither that 'you watch a movie' nor that 'you play a video game'. Since the exclusive disjunction with three disjuncts in task (1) is embedded, logical forms other than (18) can be attributed to it as well. But, in all likelihood, those logical forms will also allow deriving neither 'Wa' nor 'Pa'.

Conclusions

Several reasoning tasks seem to be challenges for classical logic. This is because the answers individuals give to them appear to be, in principle, inconsistent with the latter logic. That is the case of tasks including embedded exclusive disjunctions with three disjuncts such as (1).

According to classical logic, the answers to the questions in (1) should be positive. If sentence A in (1) is true, sentence B in (1) has to be false. However, sentence B is an exclusive disjunction. So, it is false whenever its two disjuncts are true. This means that task (1) allows a scenario in which 'you read a book', 'you watch a movie', and 'you play a video game' are true at once.

If classical logic is left and a framework such as that of the theory of mental models is assumed, the problem is solved. To note that a situation in which 'you read a book', 'you watch a movie', and 'you play a video game' is a possible scenario in (1) is not easy for human mental abilities. Given so complex tasks, human beings often only use intuition. If that occurs, only three possible scenarios are identified for (1). In one of them, 'you read a book'. In the second one, 'you watch a movie'. In the last one, 'you play a video game'. This is, from the theory of mental models, the reason for the habitual answers in tasks such as (1).

Nevertheless, it is also possible to understand what happens with task (1) from first-order predicate logic. If a well-formed formula in the latter logic capturing the logical form of (1) is built, it is possible to realize that just the formula and the information provided in the task do not enable to derive neither that 'you watch a movie' nor that 'you play a video game'. Therefore, it is impossible to positively respond to the questions in the task.

One might object that those questions do not ask whether or not 'you watch a movie' and 'you play a video game' can be derived. The questions are not about whether or not 'you watch a movie' and 'you play a video game' are the case. They only ask about the possibility for those actions to be the case. Nonetheless, the concept of possibility is not a concept of first-order predicate logic. To consider the concept of possibility and, hence, a logical operator for possibility means to move to modal logic. This is a key point, since this paper has been intended to stay within the limits of first-order predicate calculus. If those limits are respected, the majority answers in tasks such as (1) seem to be the suitable ones. From the arguments above, it can be thought that this also apply to tasks with contents different from those in (1). As accounts based on classical logic require, the explanation is not beyond formal structures.

References

Barres, P. E. & Johnson-Laird, Ph. N. (2003). On Imagining What Is True (and What is False). *Thinking and Reasoning*, *9*(1): 1-42. https://doi.org/10.1080/13546780244000097.

Byrne, R. M. J. & Johnson-Laird, Ph. N. (2020). *If* and *or*: Real and Counterfactual Possibilities in Their Truth and Probability. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *46*(4):760-780. https://doi.org/10.1037/xlm0000756.

Espino, O., Byrne, R. M. J. & Johnson-Laird, Ph. N. (2020). Possibilities and the Parallel Meanings of Factual and Counterfactual Conditionals. *Memory & Cognition*, (48): 1263-1280. https://doi.org/10.3758/s13421-020-01040-6.

Johnson-Laird, Ph. N. (2023). Possibilities and Human Reasoning. *Possibilities Studies & Society*. https://doi.org/10.1177/27538699231152731.

Johnson-Laird, Ph. N., Quelhas, A. C. & Rasga, C. (2021). The Mental Model Theory of Free Choice Permissions and Paradoxical Disjunctive Inferences. *Journal of Cognitive Psychology*, *33*(8): 951-973. https://doi.org/10.1080/20445911.2021.1967963.

Johnson-Laird, Ph. N. & Ragni, M. (2019). Possibilities as the Foundation of Reasoning. *Cognition*, (193). https://doi.org/10.1016/j.cognition.2019.04.019.

Khemlani, S., Byrne, R. M. J. & Johnson-Laird, Ph. N. (2018). Facts and Possibilities: A Model-Based Theory of Sentential Reasoning. *Cognitive Science*, Vol. 42, n. 6, pp. 1887-1924. https://doi.org/10.1111/cogs.12634.

Khemlani, S, Hinterecker, Th. & Johnson-Laird, Ph. N. (2017). The Provenance of Modal Inference. G. Gunzelmann, A. Howes, T. Tenbrink, & E. J. Davelaar (eds.), *Computational Foundations of Cognition*. Austin, TX: Cognitive Science Society: 663-668.

Khemlani, S. & Johnson-Laird, Ph. N. (2009). Disjunctive Illusory Inferences and How to Eliminate Them. *Memory & Cognition*, *37*(5): 615-623. https://doi.org/10.3758/MC.37.5.615.

Khemlani, S. & Johnson-Laird, Ph. N. (2022). Reasoning About Properties: A Computational Theory. *Psychological Review*, *129*(2): 289-312. https://doi.org/10.1037/rev0000240.

López-Astorga, M. (2014). A Formal Theory Can Explain Disjunctive Illusory Inferences. *Círculo de Lingüística Aplicada a la Comunicación*, (60): 122-143. https://doi.org/10.5209/rev_CLAC.2014.v60.47445.

(2021). *The Possibilities of the Mental Model Theory*. Curitiba, PR, Brazil: Appris Ltda.

O'Toole, R. R. & Jennings, R. E. (2004). The Megarians and the Stoics. D. M. Gabbay & J. Woods (eds.), *Handbook of the History of Logic*, *Volume 1. Greek, Indian and Arabic Logic*: 397-522. Amsterdam, Netherlands: Elsevier. https://doi.org/10.1016/S1874-5857(04)80008-6.

Quelhas, A. C., Rasga, C. & Johnson-Laird, Ph. N. (2019). The Analytic Truth and Falsity of Disjunctions. *Cognitive Science*, *43*(9). https://doi.org/10.1111/cogs.12739.